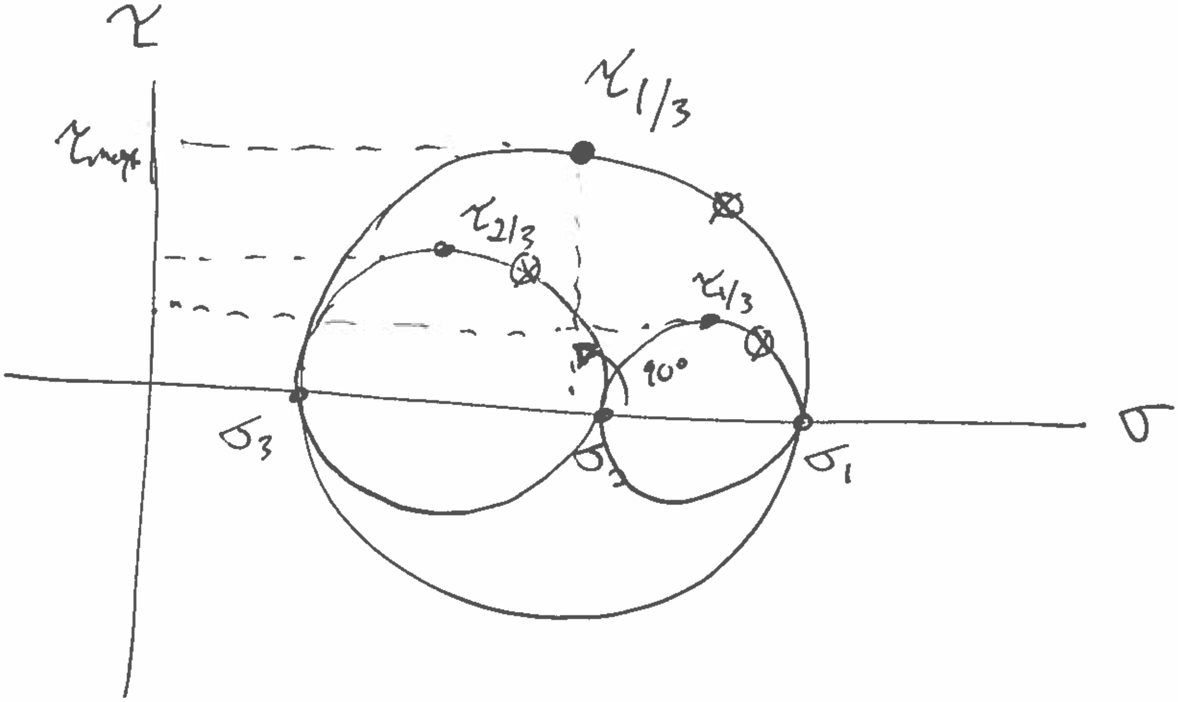
3D



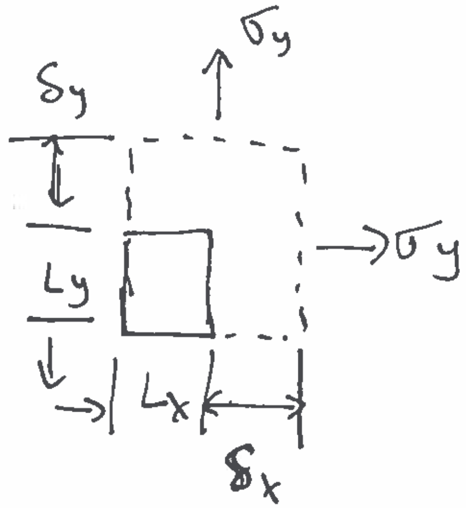
Elastic Strain

Under an applied load a structural member will deform.

stress ⇒ nearly impossible to measure

strain ⇒ measure direction and magnitude

Normal Strain

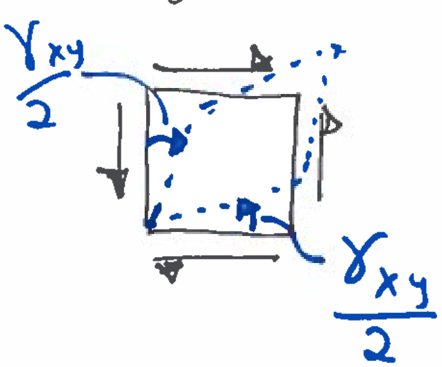


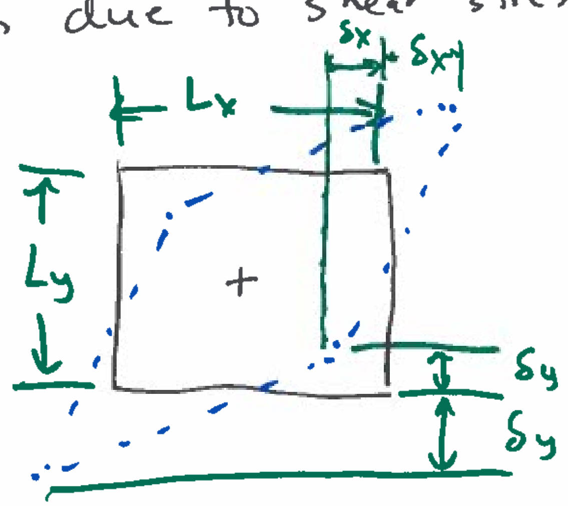
$$\epsilon_x = \frac{\delta_x}{L_x} \qquad \epsilon_y = \frac{\delta_y}{L_y}$$

$$\nu = \frac{\epsilon_x}{\epsilon_y}$$ Poisson's Ratio

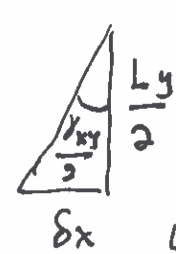
Shear Strain

Angular distortion due to shear stresses: γ





assume small angles



$$\tan(\gamma_{xy}) = \frac{2\delta_x}{L_y} + \frac{2\delta_y}{L_x}$$

$$\gamma_{xy} = \frac{2\delta_x}{L_y} + \frac{2\delta_y}{L_x} \Rightarrow \frac{\gamma_{xy}}{2} = \frac{\delta_x}{L_y} + \frac{\delta_y}{L_x}$$

<u>Strain</u> <u>Tensor</u> 3D

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_z \end{bmatrix}$$

<u>Plane</u> <u>Strain</u> (2D) <u>Principal</u> <u>Strains</u>

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\left(\frac{\gamma}{2}\right)_{max} = \frac{\epsilon_1 - \epsilon_2}{2}$$

Hooke's Law: $\sigma = E\epsilon$

modulus of elasticity

E: Mod. of elas. (Young's modulus)

G: shear modulus (modulus of rigidity)

ν: Poisson's ratio (≈ 0.3 for most structural mats.)

Table A-5

$$E = 2G(1+\nu)$$

General Form

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\gamma_{xz} = \frac{1}{G}\tau_{xz}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$

Matrix Form

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{yz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

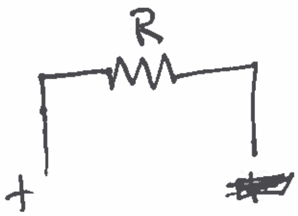
inverted

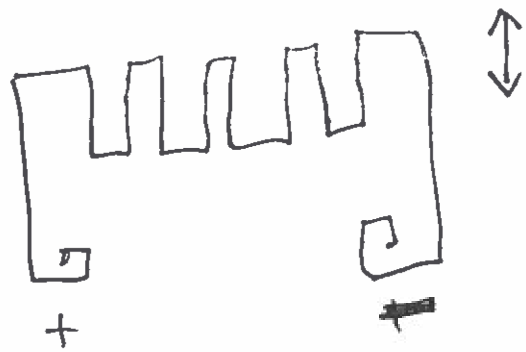
$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xx} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

Stress Type	Principal Strains	Principal Stresses
Uniaxial	$\epsilon_1 = \frac{\sigma_1}{E}$	$\sigma_1 = E\epsilon_1$
	$\epsilon_2 = -\nu\epsilon_1$	$\sigma_2 = 0$
	$\epsilon_3 = -\nu\epsilon_1$	$\sigma_3 = 0$
Biaxial	$\epsilon_1 = \frac{\sigma_1}{E} - \frac{\nu\sigma_2}{E}$	$\sigma_1 = \frac{E(\epsilon_1 + \nu\epsilon_2)}{1-\nu^2}$
	$\epsilon_2 = \frac{\sigma_2}{E} - \frac{\nu\sigma_1}{E}$	$\sigma_2 = \frac{E(\epsilon_2 + \nu\epsilon_1)}{1-\nu^2}$
	$\epsilon_3 = \frac{-\nu\sigma_1}{E} - \frac{\nu\sigma_2}{E}$	$\sigma_3 = 0$
Triaxial	$\epsilon_1 = \frac{\sigma_1}{E} - \frac{\nu\sigma_2}{E} - \frac{\nu\sigma_3}{E}$	$\sigma_1 = \frac{E\epsilon_1(1-\nu) + \nu E(\epsilon_2 + \epsilon_3)}{1-\nu-2\nu^2}$
	$\epsilon_2 = \frac{\sigma_2}{E} - \frac{\nu\sigma_1}{E} - \frac{\nu\sigma_3}{E}$	$\sigma_2 = \frac{E\epsilon_2(1-\nu) + \nu E(\epsilon_1 + \epsilon_3)}{1-\nu-2\nu^2}$
	$\epsilon_3 = \frac{\sigma_3}{E} - \frac{\nu\sigma_1}{E} - \frac{\nu\sigma_2}{E}$	$\sigma_3 = \frac{E\epsilon_3(1-\nu) + \nu E(\epsilon_1 + \epsilon_2}{1-\nu-2\nu^2}$

Strain Gages

- change in electrical resistance is proportional to strain
- measure plane strain on surface of an object





Gauge Factor

$$GF = \frac{\Delta R / R_G}{\epsilon}$$

ΔR: change in resistance

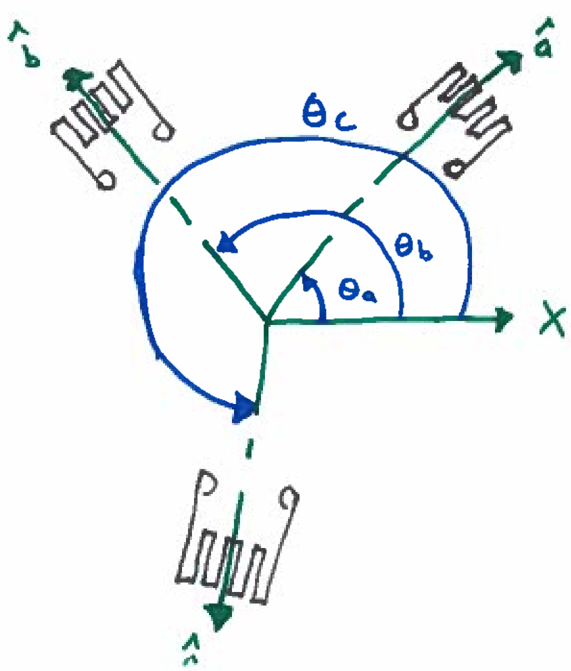
R_G: nominal resistance

ϵ: strain

$$GF \approx 2$$

Strain Gage Rosettes

- common layout to measure all components of plane strain
- each strain gage measures normal strain on surface along the $\hat{a}$, $\hat{b}$, or $\hat{c}$ direction



These equations give the relationship between strain along $\hat{a}, \hat{b}$, and $\hat{c}$ to the standard components of strain along a desired coordinate system $\hat{i}\hat{j}\hat{k}$.

$$\epsilon_a = \epsilon_x \cos^2\theta_a + \epsilon_y \sin^2\theta_a + \gamma_{xy} \sin\theta_a \cos\theta_a$$

$$\epsilon_b = \epsilon_x \cos^2\theta_b + \epsilon_y \sin^2\theta_b + \gamma_{xy} \sin\theta_b \cos\theta_b$$

$$\epsilon_c = \epsilon_x \cos^2\theta_c + \epsilon_y \sin^2\theta_c + \gamma_{xy} \sin\theta_c \cos\theta_c$$

These equations can be solved for ϵ_x, ϵ_y and γ_{xy}

$$\begin{bmatrix} \epsilon_a \\ \epsilon_b \\ \epsilon_c \end{bmatrix} = \begin{bmatrix} \cos^2\theta_a & \sin^2\theta_a & \sin\theta_a \cos\theta_a \\ \cos^2\theta_b & \sin^2\theta_b & \sin\theta_b \cos\theta_b \\ \cos^2\theta_c & \sin^2\theta_c & \sin\theta_c \cos\theta_c \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Solve using Gaussian Elimination

For specific angles $\theta_a, \theta_b, \theta_c$ the solution simplifies.

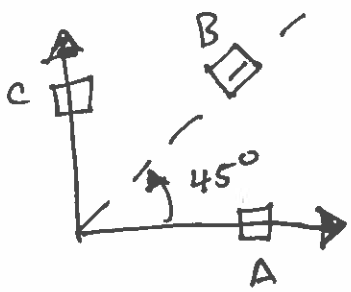
For example if $\theta_a = 0°$, $\theta_b = 45°$, and $\theta_c = 90°$ then

$$\epsilon_x = \epsilon_a$$

$$\epsilon_y = \epsilon_c$$

$$\gamma_{xy} = 2\epsilon_b - (\epsilon_a + \epsilon_c)$$

Example: Strain Gages and Principal Stresses



$E = 200$ GPa

$\nu = 0.3$

What are the principal stresses if the strain gage readings are $\epsilon_a = 60E\text{-}6$, $\epsilon_b = -75E\text{-}6$, $\epsilon_c = 232E\text{-}6$?

State of Strain

$\epsilon_x = 60E\text{-}6$

$\epsilon_y = 232E\text{-}6$

$\gamma_{xy} = 2(-75E\text{-}6) - (60E\text{-}6 + 232E\text{-}6) = -442E\text{-}6$

Principal Strains

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$\epsilon_1 = 382E\text{-}6$

$\epsilon_2 = -91E\text{-}6$

Principal Stresses

$$\sigma_1 = \frac{E(\epsilon_1 + \nu\epsilon_2)}{1-\nu^2} = 78 \text{ MPa}$$

$$\sigma_2 = \frac{E(\epsilon_2 + \nu\epsilon_1)}{1-\nu^2} = 5 \text{ MPa}$$