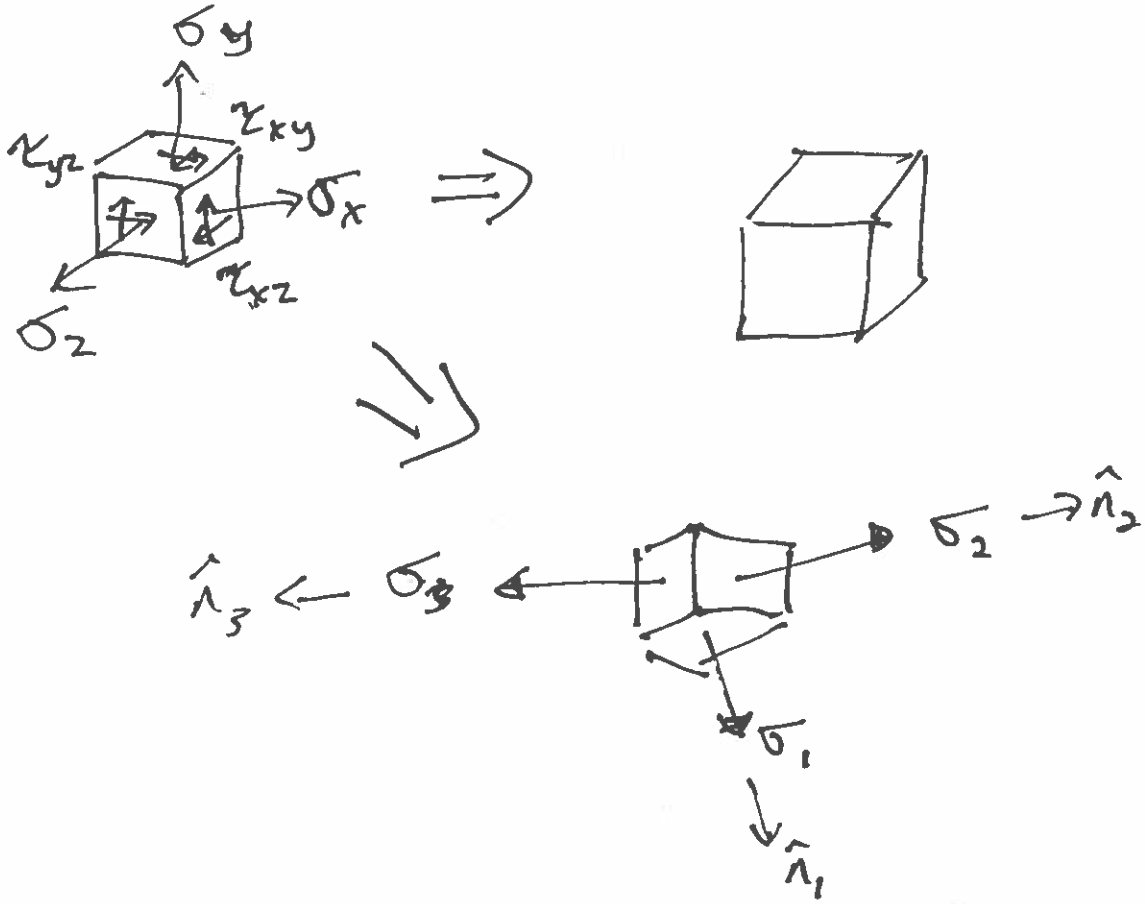
LECTURE 9

Monday, October 10, 2016

Principal Stresses

From Theory of Elasaty:

At any point with a general state of stress, an element can be oriented such that the shear components vanish.



$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \overset{?}{\Longrightarrow} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{bmatrix}$$

principal stresses: $\sigma_1 \geq \sigma_2 \geq \sigma_3$

principal directions: $\hat{n}_1 \quad \hat{n}_2 \quad \hat{n}_3$

unknown principal stress

$$\bar{\sigma}_n = \sigma \hat{n} = \sigma \left(l \hat{i} + m \hat{j} + n \hat{k} \right)$$

unknowns?

$$\bar{\sigma}_n = l \bar{\sigma}_x + m \bar{\sigma}_y + n \bar{\sigma}_z$$

$$= l \left(\sigma_x \hat{i} + \tau_{xy} \hat{j} + \tau_{xz} \hat{k} \right)$$
$$+ m \left(\tau_{yx} \hat{i} + \sigma_y \hat{j} + \tau_{yz} \hat{k} \right)$$
$$+ n \left(\tau_{zx} \hat{i} + \tau_{zy} \hat{j} + \sigma_z \hat{k} \right)$$

$$l\sigma = l\sigma_x + m\tau_{yx} + n\tau_{zx}$$

$$m\sigma = l\tau_{xy} + m\sigma_y + n\tau_{yz}$$

$$n\sigma = l\tau_{zx} + m\tau_{yz} + n\sigma_z$$

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{\underline{3 \times 3}}$

$$A\bar{V} = \lambda \bar{V}$$

$$(A - \lambda I)\bar{V} = 0$$

unknowns

$$(\sigma_{ij} - \sigma I)\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

Eigenvalue and eigenvector problem.

$\det|A| = 0 \Rightarrow$ above is true only if $\det|A| = 0$

$$\underbrace{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0}$$

cubic polynomial in σ
characteristic equation

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \det\begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \det\begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix} + \det\begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix}$$

$$I_3 = \underset{3\times 3}{\det\begin{vmatrix} \sigma_{ij} \end{vmatrix}} \qquad \sigma_{ij} = \begin{bmatrix} \sigma_x & & \vdots \\ & \sigma_y & \\ \vdots & & \sigma_z \end{bmatrix}$$

3 solutions in general

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$ order is convention

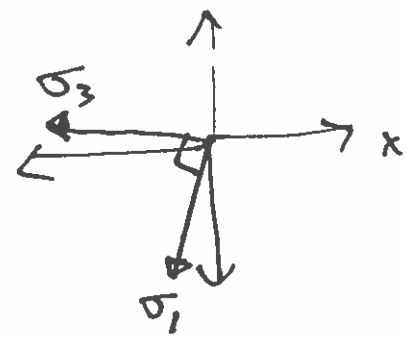
Find principal direction

$$(\sigma_{ij} - \sigma I)\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

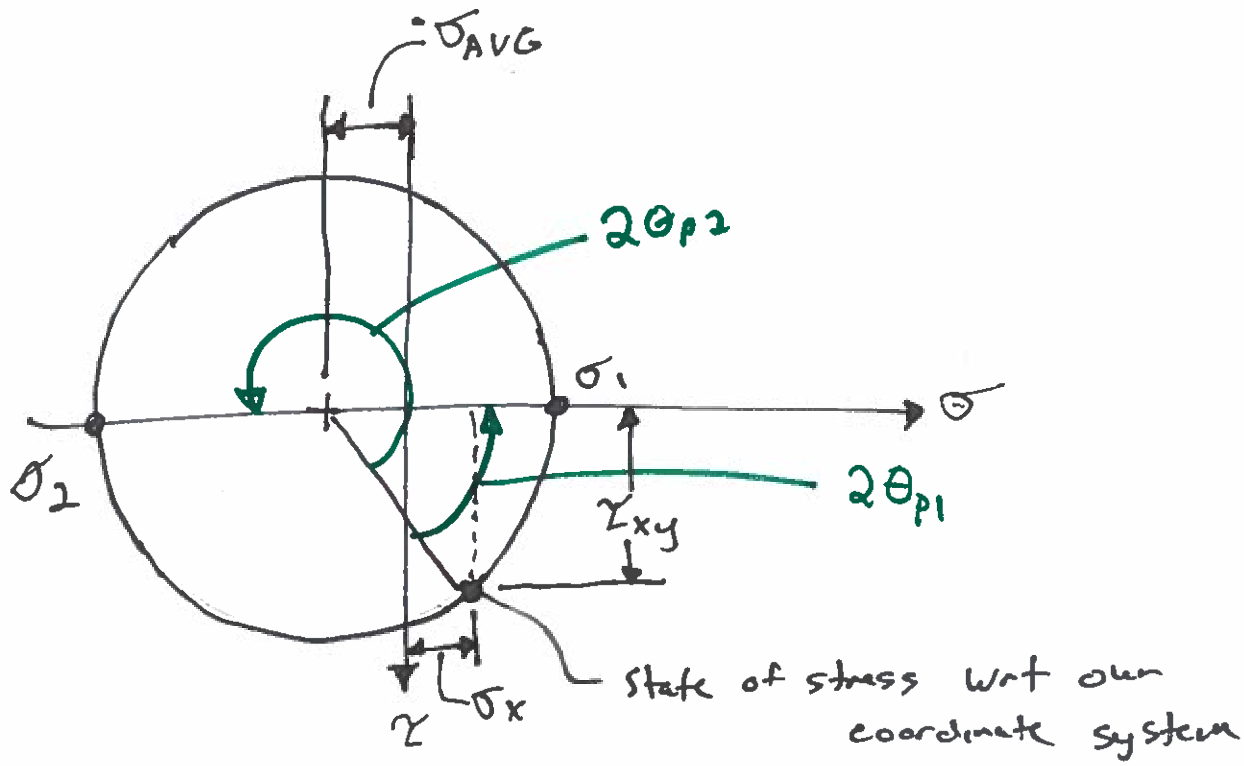
eigenvectors are principal directions

$$\hat{n}_1 = \underline{l_1}\,\hat{i} + \underline{m_1}\,\hat{j} + \underline{n_1}\,\hat{k}$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$



Given σ_{AVG}, σ_x, and τ_{xy} find the radius and the principal stresses of the following Mohr's circle.

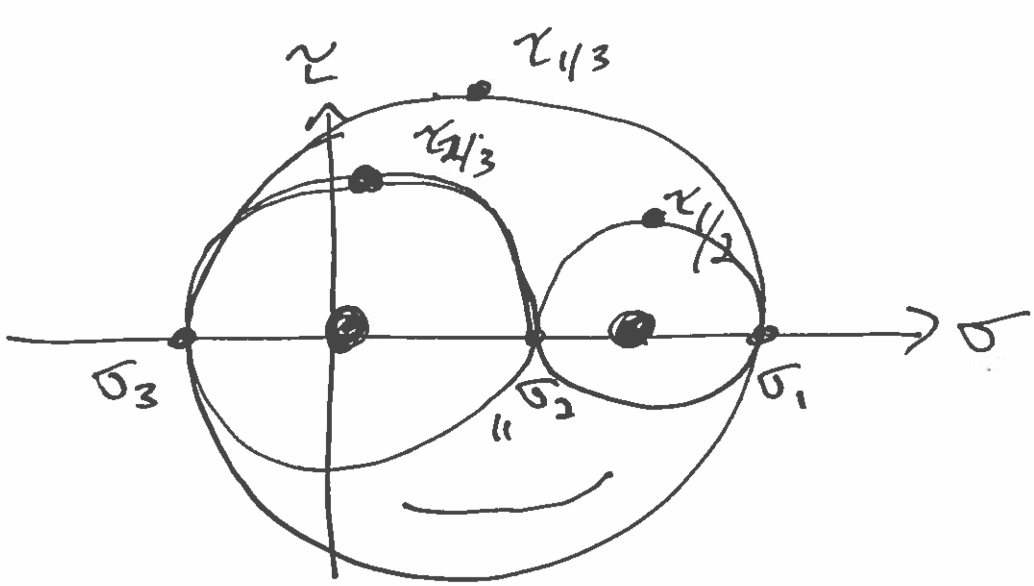


$$R = \sqrt{(\sigma_x - \sigma_{AVG})^2 + \tau_{xy}^2} \qquad \sigma_{AVG} = \frac{\sigma_x + \sigma_y}{2}$$

$$\sigma_1 = R - \sigma_{AVG}$$

$$\sigma_2 = \sigma_{AVG} - R$$

3D



$\tau_{max} = \tau_{1/3}$

$\tau_{1/3} = \dfrac{\sigma_1 - \sigma_3}{2}$

$\tau_{2/2} = \dfrac{\sigma_1 - \sigma_2}{2}$

$\tau_{2/3} = \dfrac{\sigma_2 - \sigma_3}{2}$

Cubic Eq:

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)\sigma - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{xz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2) = 0$$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$