EME 150A LECTURE 14 Friday, October 21, 2016

Deformation and Stiffness

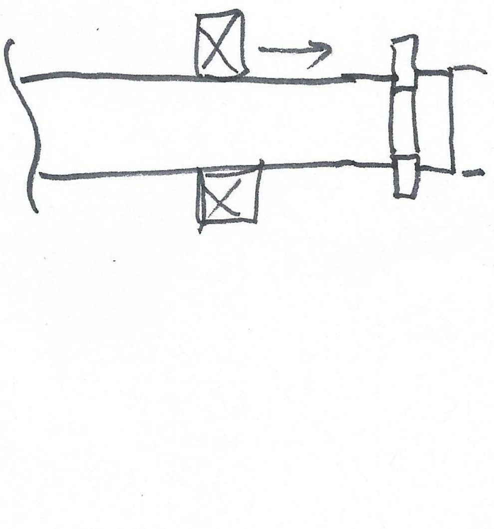
Chap 4

Design for high rigidity

- minimize misalignment
- avoid interfence w/ other components
- reduse noise
- reduce wear rates
- reduce stress

Design for flexibility

- energy storage and absorbtion
  - springs
- elastic deformations for change in dimensions



snap ring

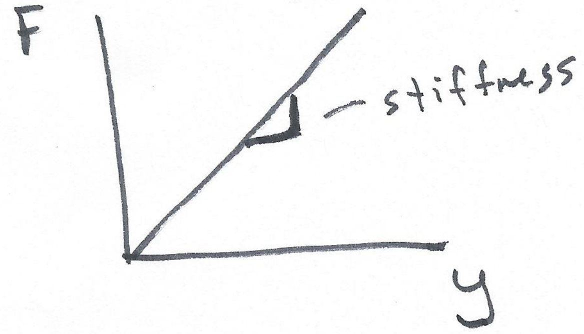
Rigidity

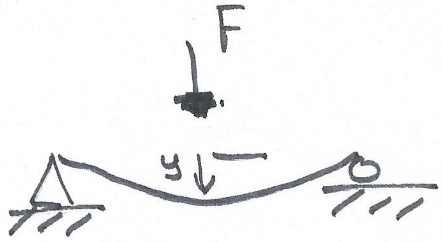
deflection per load $\frac{\Delta y}{F}$

- mod of Elasticity gives a good indication of rigidity
- geometry of the structural element is essential to characterize rigidity
- inverse of rigidity is "stiffness"

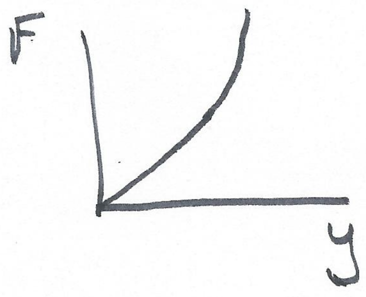
$$k = \frac{F}{\Delta y}$$

Linear Springs

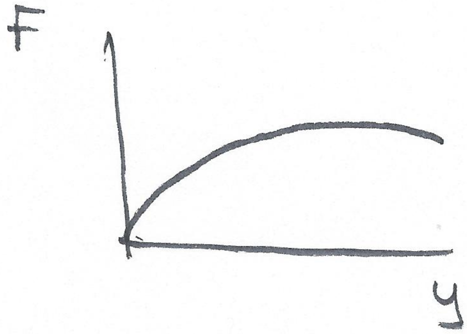




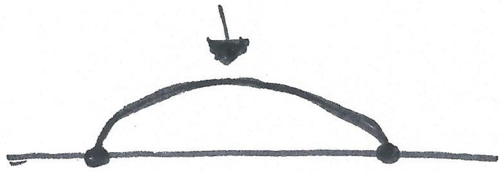
Nonlinear Springs



stiffening spring



Softening spring



Stiffness (spring constant) for linear springs

Axial

$$\delta = \frac{F\ell}{AE} \qquad k = \frac{AE}{\ell}$$

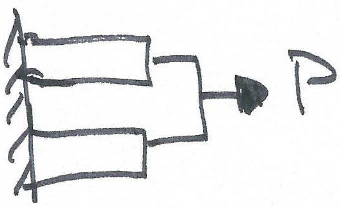
stiffness

Torsional

$$\theta = \frac{T\ell}{GJ} \qquad k = \frac{GJ}{\ell}$$

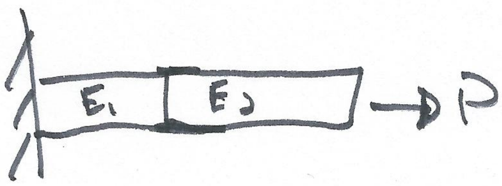
torsional rigidity

Parallel springs



$$k = \sum_{i=1}^{N} k_i$$

Series



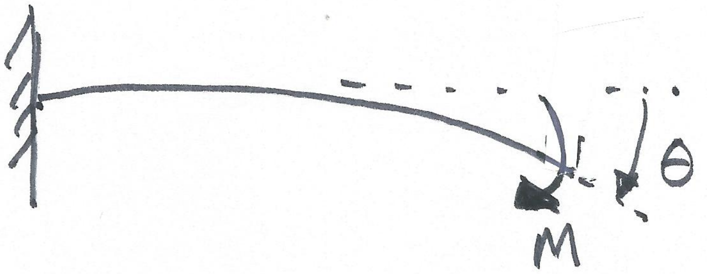
$$k = \frac{1}{\sum_{i=1}^{N} \frac{1}{k_i}}$$

stiffness due to

Moment that cause angular def.

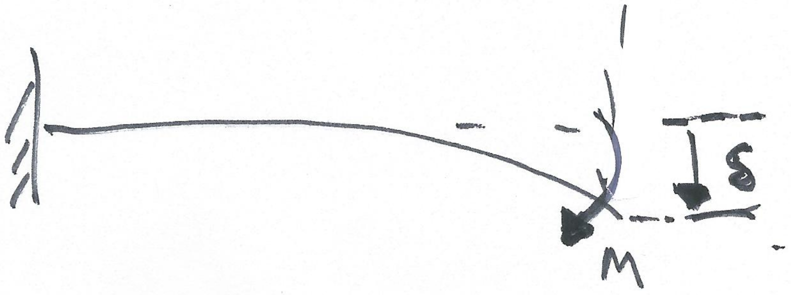
Bending (angular stiffness)

flexural rigidity



$$\Theta = \frac{ML}{EI} \qquad k = \frac{EI}{L}$$

Bending (linear stiffness)



$$\delta = \frac{ML^2}{2EI} \qquad k = \frac{2EI}{L^2}$$

Bending (linear stiffness wrt Force)

$$\delta = \frac{PL^3}{3EI} \qquad k = \frac{3EI}{L^3}$$

Relative Stiffness

① 1" OD round steel bar, 10" w/ tensile load

$$k = \frac{AE}{l} \Rightarrow k = 23.6 \times 10^5 \ \frac{lb}{in}$$

② same element w/ bending

$$K = \frac{3EI}{l^3} \Rightarrow k = 4.4 \times 10^3 \ \frac{lb}{in}$$

③ cantilever w/ shear load

$$K = \frac{9AG}{10l} \Rightarrow k = 81.3 \times 10^4 \ \frac{lb}{in}$$

Same bar is most stiff axially, then in shear, and then in bending

④ torsion (solid bar)

$$k_t = \frac{JG}{l} \Rightarrow k_t = 1.13 \times 10^5 \ \frac{lb\,in}{rad}$$

⑤ tube w/ circular cross section same area as the bar in ①

$$k_t = \frac{JG}{l} \Rightarrow k_t = 55.3 \times 10^5 \ \frac{lb \cdot in}{rad}$$

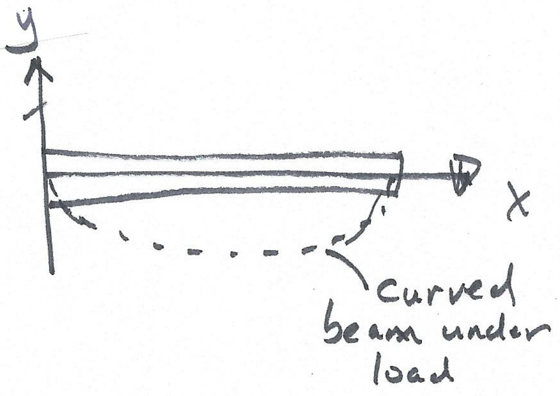
tube with same amount of material is 55X stiffer

Deflection of Beams

theory of elasticity

Curvature: ρ

$$\frac{1}{\rho} = \frac{M}{EI}$$



From math: rho

$$\frac{1}{\rho} = \frac{d^2y/d^2x}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

$\frac{dy}{dx}$ is small

$$\frac{dy}{dx} \approx 0$$

$$\Downarrow$$

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

$$\boxed{\frac{M}{EI} = \frac{d^2y}{dx^2}}$$

$$\frac{V}{EI} = \frac{d^3y}{dx^3} \qquad \frac{q}{EI} = \frac{d^4y}{dx^4}$$

Given $q(x)$

$$V = \int q(x)\,dx + C_V$$
$$M = \int V(x)\,dx + C_m$$

C_V and C_m obtained from static equilibrium conditions

$$EI\theta = \int M(x)\,dx + C_1$$
$$y = \int \theta(x)\,dx + C_2$$

$C_1 + C_2$ obtained from boundary conditions

Multiple Ways to solve these

- method of sections
- Superposition *
- moment area method
- singularity functions *
- numerically integrate

Superposition

- Many simple beam cases have been pre-solved and the results can be algebraically combined

- Table A-9 ⇒ look for small set of beams

- Roark's Formulas for Stress and Strain

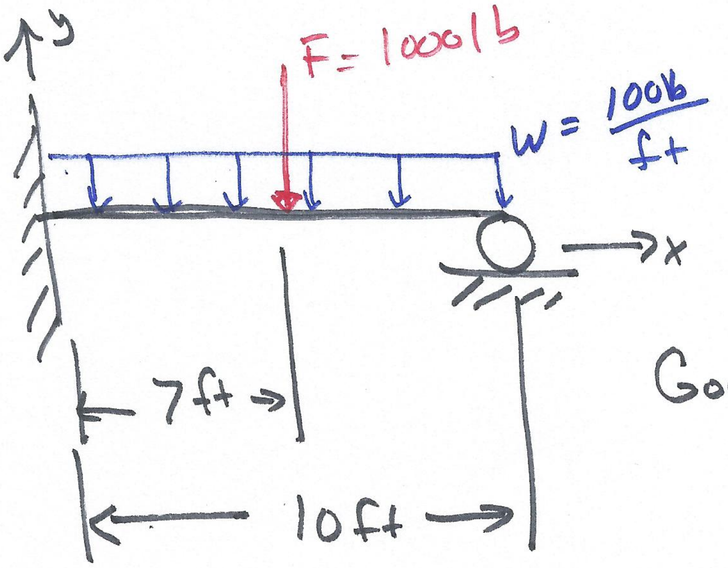
Rules

- each effect must be linearly related to the load that produced it, e.g. $\delta = \frac{PL}{AE}$

- a load does not create a condition that affect the result of another load

- deformation from any specific load is not large enough to appreciably alter the geometric relation of the structural system

Example: Superposition



$E = 30E6$ psi

$I = 5$ in^4

Goal: What is y_{max}?

From Table A-9 #12

$$y_{AB} = \frac{Fbx^2}{12EI\ell^3}\left[3\ell(b^2-\ell^2) + x(3\ell^2-b^2)\right]$$

$$y_{BC} = y_{AB} - \frac{F(x-a)^3}{6EI}$$

#13

$$y = \frac{wx^2}{48EI}(\ell-x)(2x-3\ell)$$

$$\left.\begin{aligned} y_{tot\,AB} &= y_{AB} + y \\ y_{tot\,BC} &= y_{BC} + y \end{aligned}\right\} \text{superposition}$$

$$y_{tot\,AB} = C_1 x^4 + C_2 x^3 + C_3 x^2 + C_4 x^1 + C_5$$

$$\frac{dy_{tot\,AB}}{dx} = 0$$ Solve for x where $\theta = 0$

cubic polynomial

See the Jupyter Notebook with code to solve for the roots of this polynomial.

* Note: determine if max deflection is between $0 \le x \le 7$ or $7 \le x \le 10$ *

$$\theta_{AB}(x) = -9.26\times10^{-9}x^3 + 3.54\times10^{-6}x^2 - 2.1\times10^{-4}x$$

$$\theta_{BC}(x) = -9.26\times10^{-9}x^3 + 2.05\times10^{-7}x^2 + 3.5\times10^{-4}x - 0.024$$

$\theta_{BC} = 0 \Rightarrow$ no solutions between 7ft and 10 ft

$x = (0,\ \underline{73.11},\ 309.03)$ in for $\theta_{AB}(x) = 0$

only valid answer

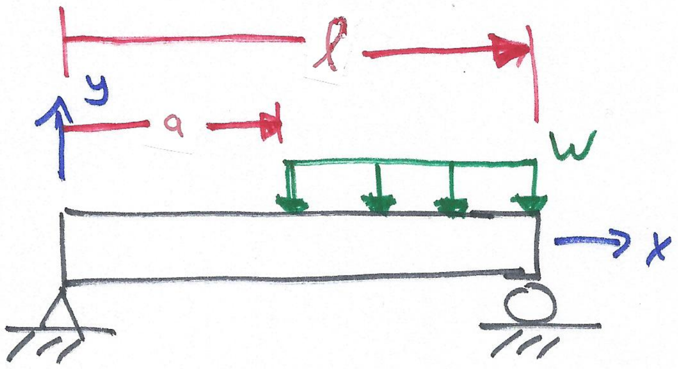
$$y_{AB}(73.11) = \boxed{y_{max} = -0.102''}$$

Deflection of Beam : Singularity Functions

$$EI\Theta = \int M(x)\,dx + c_1 \Rightarrow \Theta = \frac{1}{EI}\int M(x)\,dx + c_1$$

$$y = \int \Theta(x)\,dx + C_2 = \frac{1}{EI}\int\int M(x)\,dx + C_3$$

<u>Example</u>



<u>Find</u>

1) expressions for slope and deflection and also 2) max deflection with following parameters

$l = 10''$

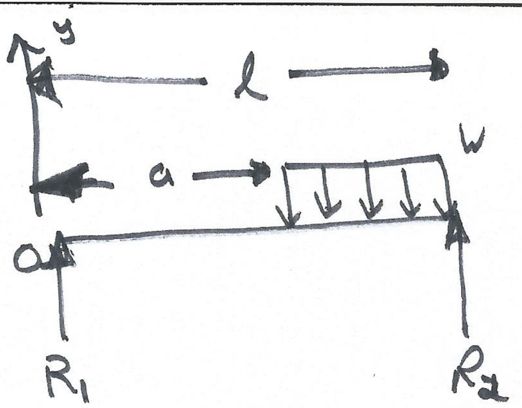
$a = 4''$

$w = 100\,\frac{lb}{in}$

$E = 30$ Mpsi

$I = .163\ in^4$

FBD



$$q(x) = R_1\langle x\rangle^{-1} - w\langle x-a\rangle^{0} + w\langle x-l\rangle^{0} + R_2\langle x-l\rangle^{-1}$$

$$V(x) = R_1\langle x\rangle^{0} - w\langle x-a\rangle^{1} + w\langle x-l\rangle^{1} + R_2\langle x-l\rangle^{0}$$

$$M(x) = R_1\langle x\rangle^{1} - \frac{w}{2}\langle x-a\rangle^{2} + \frac{w}{2}\langle x-l\rangle^{2} + R_2\langle x-l\rangle^{1}$$

$$V(l^+) = 0 = R_1 - w(l^+-a) + w(l^+-l) + R_2$$

$$M(l^+) = 0 = R_1 l^+ - \frac{w}{2}(l^+-a)^2 + \frac{w}{2}(\underbrace{\cancel{l^+-l}}_{\to 0 \text{ small}})^2 + R_2(l^+-l)^1$$

$$\underline{R_1 = \frac{w}{2l}(l-a)^2}$$

$$0 = \frac{w}{2l}(l-a)^2 - w(l-a) + R_2$$

$$\underline{R_2 = w(l-a) - \frac{w}{2l}(l-a)^2}$$

$R_1 = 180\text{ lb}$ $\quad R_2 = 420\text{ lb}$

* drop discontinuties @ end of beam *

$$EI\theta = \frac{R_1}{2}\langle x\rangle^2 - \frac{W}{6}\langle x-a\rangle^3 + C_1$$

$\theta = \frac{1}{EI}$

$$EIy = \frac{R_1}{6}\langle x\rangle^3 - \frac{W}{24}\langle x-a\rangle^4 + C_1x + C_2$$

$y(0) = 0$ $y(l) = 0$ no deflection @ simply supported ends

boundary conditions

@ $x=0$: $0 = C_2$

@ $x=l$: $0 = \frac{R_1 l^3}{6} - \frac{W}{24}(l-a)^4 + C_1 l$

Solve for C_1 and C_2

$$C_1 = \frac{W}{24l}(l-a)^4 - \frac{R_1 l^2}{6}$$

$$C_2 = 0$$

1) Expressions for θ and y:

$$\theta = \frac{1}{EI}\left[\frac{R_1}{2}\langle x\rangle^2 - \frac{W}{6}\langle x-a\rangle^3 + \frac{W}{24l}(l-a)^4 - \frac{R_1 l^2}{6}\right]$$

$$y = \frac{1}{EI}\left[\frac{R_1}{6}\langle x\rangle^3 - \frac{W}{24}\langle x-a\rangle^4 + \frac{Wx}{24l}(l-a)^4 - \frac{R_1 l^2 x}{6}\right]$$

The max deflection will be where $a < x < l$, so:

$$\Theta(x) = \frac{1}{EI}\left[\frac{R_1}{2}x^2 - \frac{W}{6}(x-a)^3 + \frac{W}{24l}(l-a)^4 - \frac{R_1 l^2}{6}\right]$$

Where is $\Theta(x) = 0$? Solve cubic polynomial for x.

$$-3.4E{-6}\,x^3 + 5.93E{-5}\,x^2 - 1.64E{-4}\,x - 2.45E{-4} = 0$$

$$x = (-1.19,\ \underbrace{5.26}_{},\ 13.3)$$

$4'' < 5.26'' < 10''$ only valid answer

2) max deflection

$y(5.26) = -0.00176''$