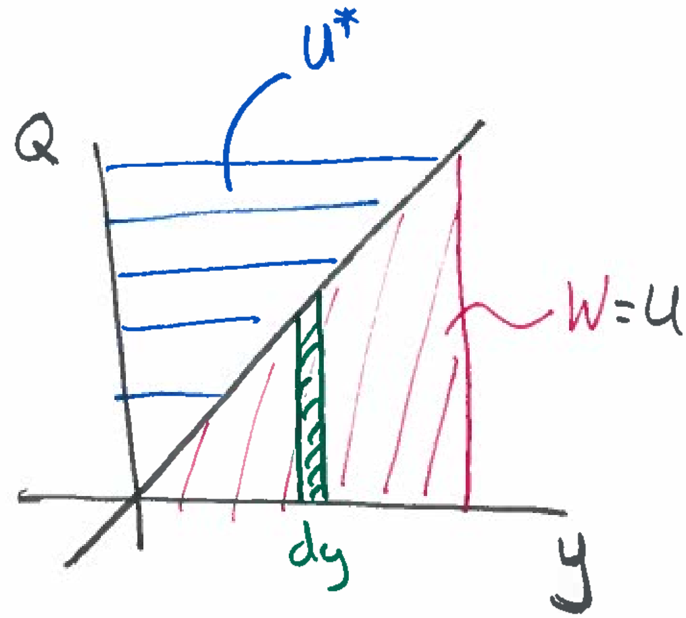
Strain Energy



Workdone on the material by a load Q to achieve a deflection y.

$$W = \int Q(y)\, dy$$

Work causes potential energy to be stored in the material ⇒ "strain energy"

$U \equiv$ Strain energy

$W = U$

U^* complementary strain energy

$$U^* = \int y(Q)\, dQ$$

For linear stiffness: $dU^* = dU = y\, dQ$

$U = U^*$ $\boxed{y = \frac{dU}{dQ}}$ valid for a single applied load on a structural element

If element is subjected to multiple loads, Q_i, all within elastic range.

The the deflection, y_i, associated with the point of application of Q_i is:

$$y_i = \frac{\partial U}{\partial Q_i} \Rightarrow \text{Castigliano's Therom}$$

U: total strain energy of the structure

Q_i: single applied load

y_i: deflection at the point application of Q_i in the direction of Q_i

Castigliano's Theorem:

When an element is elastically deflected by any combination of loads, the deflection at any point, in any direction is equal to the partial derivative of the total strain energy wrt load at that point in that direction.

* The applied load may or may not exist.

$$U = U^* = \int y(Q)\,dQ = \int \frac{PL}{AE}\,dP = \frac{P^2L}{2AE}$$

axial loaded

$$\delta = \frac{PL}{AE}$$

Load Type	Factors	Strain Energy constant factors	Strain energy variable factors
Axial	A, E, P	$U = \frac{P^2L}{2EA}$	$U = \int_0^L \frac{P^2}{2AE}\,dx$
Bending	I, E, M	$U = \frac{M^2L}{2EI}$	$U = \int_0^L \frac{M^2}{2EI}\,dx$
Torsion	J, G, T	$U = \frac{T^2L}{2GJ}$	$U = \int_0^L \frac{T^2}{2GJ}\,dx$
Shear	A, G, V	$U = \frac{CV^2L}{2AG}$	$U = \int_0^L \frac{CV^2}{2AG}\,dx$

Table 4-1 C

Cross Sections	C
rectangular	1.2
circular	1.11
thin round tube	2.00
box sec	1.00
etc	

Example Find the deflection at the midpoint using Castigliano's Theorem.

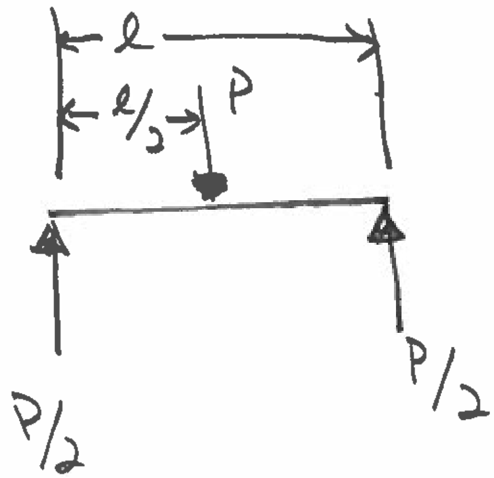
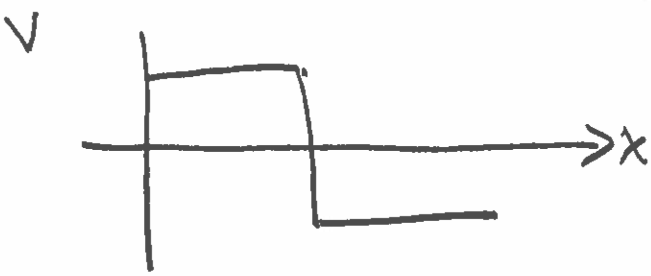
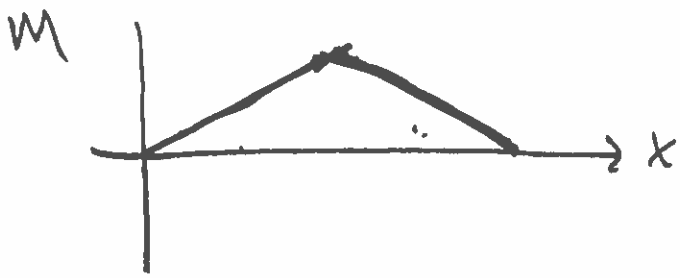


Table A-9-5

$$V = \begin{cases} P/2 & 0 \le x \le \frac{l}{2} \\ -P/2 & \frac{l}{2} \le x \le l \end{cases}$$



$$M = \begin{cases} \frac{P}{2}x & 0 \le x \le \frac{l}{2} \\ \frac{Pl}{2} - \frac{Px}{2} & \frac{l}{2} \le x \le l \end{cases}$$



cross section

Bending Strain Energy

$$U_{①} = \int_0^{l/2} \frac{M^2}{2EI}\,dx = U_{②} = \int_{l/2}^{l} \frac{M^2}{2EI}\,dx$$

Shear Strain Energy

$$U_{③} = \int_0^{l} \frac{CV^2}{2AG}\,dx$$

$C = 1.2 \Rightarrow$ rectangular
Table 4-1

Total Strain Energy

$$U = U_{1} + U_{2} + U_{3}$$

$$= 2\int_0^{l/2} \frac{M^2}{2EI}\,dx + \int_0^{l} \frac{CV^2}{2AG}\,dx$$

$$U = \frac{P^2L^3}{96EI} + \frac{15P^2L}{AG}$$

use Castigliano's thereom:

$$y\left(\frac{l}{2}\right) = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \frac{30PL}{AG}$$

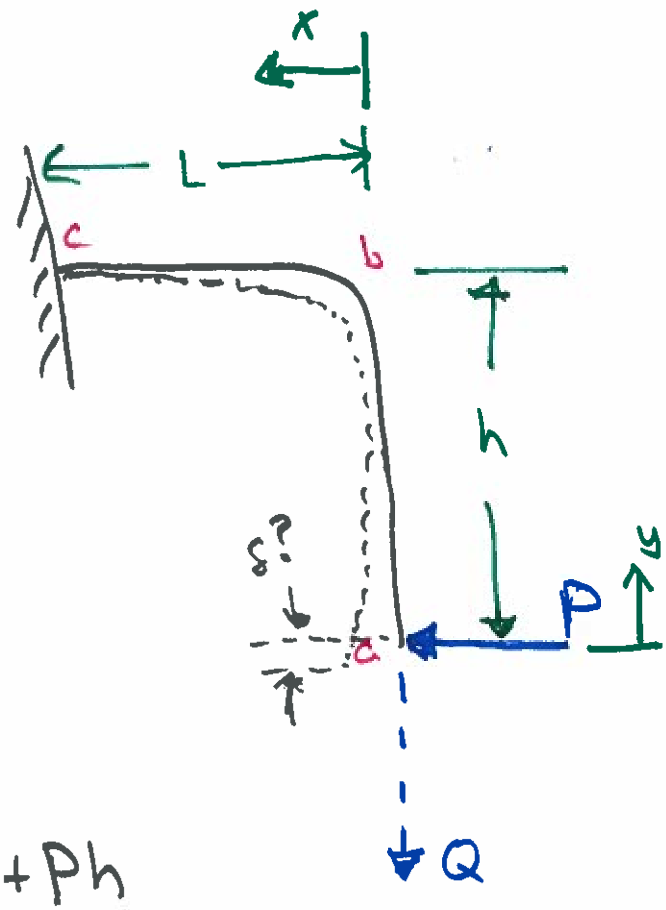
act @ $l/2$

$l/h > 10 \Rightarrow$ shear can be neglected

compare result to Table A-9-5

Example

Find the vertical deflection at the free end.



Ignore transverse shear.

1) Bending in ab, $M_{ab} = Py$
2) Bending in bc, $M_{bc} = Qx + Ph$
3) Tension in ab, Q (constant)
4) compression in cb, P (constant)

$$U = \int_0^h \frac{M_{ab}^2}{2EI}\,dy + \int_0^L \frac{M_{bc}^2}{2EI}\,dx + \frac{Q^2 h}{2EA} + \frac{P^2 L}{2EA}$$

$$U = \frac{P^2 h^3}{6EI} + \frac{Q^2 L^3}{6EI} + \frac{PQhL^2}{2EI} + \frac{P^2 h^2 L}{2EI} + \frac{Q^2 h}{2EA} + \frac{P^2 L}{2EA}$$

$$\delta_y = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} = 0 + 0 + \frac{PhL^2}{2EI} + 0 + 0 + 0$$

fictious force

$$\boxed{\delta_y = \frac{PhL^2}{2EI} \quad \text{in direction of } Q}$$

$$\delta_x = \frac{P}{EI}\left(h^3 + h^2 + L + \frac{I}{A}L\right) \quad \text{direction of } P$$