generalized coordinates $\bar{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ ($n \times 1$)

n: # GCs

$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ ($n \times 1$) generalized speeds

$\bar{X} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ u_1 \\ \vdots \\ u_n \end{bmatrix}$ ($2n \times 1$) State vector

__Non-linear__

EoM: $0 = \bar{f}(\bar{q}, \bar{u}, \dot{\bar{u}}, t)$

$0 = M\dot{\bar{u}} - F(\bar{q}, \bar{u}, t)$

solve for $\dot{\bar{u}}$ to put in explicit first order form

$M\dot{\bar{u}} = F \Rightarrow \boxed{\begin{array}{l} \dot{\bar{u}} = M^{-1}F \\ \dot{\bar{q}} = \bar{u} \end{array}}$ full explicit equations of motion

$$\dot{\bar{X}} = \begin{bmatrix} \dot{\bar{q}} \\ \dot{\bar{u}} \end{bmatrix} = \begin{bmatrix} I \\ M^{-1}F \end{bmatrix}$$

__Linear EoMs__

$$0 = \bar{f}(\bar{q}, \bar{u}, \dot{\bar{u}}, t)$$

Use a Taylor series (first two terms)

linearize about Eq.

$$\begin{bmatrix} \bar{q}_{eq} \\ \bar{u}_{eq} \end{bmatrix} = \bar{X}_{eq} \qquad \bar{V}_{eq} = \begin{bmatrix} \bar{q}_{eq} \\ \bar{u}_{eq} \\ \dot{\bar{u}}_{eq} \end{bmatrix} \qquad \bar{V} = \begin{bmatrix} \bar{q} \\ \bar{u} \\ \dot{\bar{u}} \end{bmatrix}$$

$$0 = \bar{f}(\bar{q}_{eq}, \bar{u}_{eq}, \dot{\bar{u}}_{eq}) + J_{\bar{f}}(\bar{q}_{eq}, \bar{u}_{eq}, \dot{\bar{u}}_{eq})($$

$$0 = \bar{f}(\bar{V}_{eq}) + J_{\bar{f}}(\bar{V}_{eq})(\bar{V} - \bar{V}_{eq}) = \bar{g}$$

$3n \times 3n$

$$J_{\bar{f}} = \begin{bmatrix} \frac{\partial \bar{f}}{\partial q_1} & \cdots & \frac{\partial \bar{f}}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \bar{f}}{\partial \dot{u}_1} & \cdots & \frac{\partial \bar{f}}{\partial \dot{u}_n} \end{bmatrix}$$

$$0 = \underset{n \times n}{M}\dot{\bar{u}} + \underset{n \times n}{C}\bar{u} + \underset{n \times n}{K}\bar{q}$$

$$J_{\bar{g}}^{\dot{\bar{u}}} = M$$

$$J_{\bar{g}}^{\bar{u}} = C$$

$$J_{\bar{g}}^{\bar{q}} = K$$

$M\ddot{\bar{x}} + K^{-} = 0$ $\qquad \bar{x} = \begin{bmatrix} \theta \\ \phi \end{bmatrix}$ $\qquad$ M: symmetric & positive definite

$M = \begin{bmatrix} l^2 m_p + r^2 m_b & -m_b r^2 \\ -m_b r^2 & m_b r^2 \end{bmatrix}$ mass matrix $\qquad$ Symmetric matrices

$K = \begin{bmatrix} g l m_p & g m_b r \\ g m_b r & g m_b r \end{bmatrix}$ stiffness matrix $\qquad$ K : symmetric

positive definite : $\bar{x}^T M \bar{x} > 0$ for any $\bar{q}$

$M = L L^T$ where L is the Cholesky Decomposition and it is a lower triangular matrix

$\bar{x} = (L^T)^{-1} \bar{q}$

$$M (L^T)^{-1} \ddot{\bar{q}} + K (L^T)^{-1} \bar{q} = 0$$

$$L^{-1} M (L^T)^{-1} \ddot{\bar{q}} + L^{-1} K (L^T)^{-1} \bar{q} = 0$$

$$I \ddot{\bar{q}} + \tilde{K} \bar{q} = 0$$ mass normalized Form

$\bar{q} = \bar{q}_0 \sin \omega t$

$\dot{\bar{q}} \quad \omega \quad \omega t$

$\ddot{\bar{q}} = -\omega^2 \bar{q}_0 \sin \omega t$

$$I(-\omega^2 \bar{q}_0 \sin \omega t) + \tilde{K}(\bar{q}_0 \sin \omega t) = 0$$

How can this be equal to zero?

$$(-\omega^2 I + \tilde{K}) \bar{q}_0 \sin \omega t = 0$$

$-\omega^2 I + \tilde{K} = 0$ $\qquad$ can't be zero

$\bar{q}_0 = 0$ $\qquad$ trivial case

$\sin \omega t = 0$ $\qquad$ not for time

$$\boxed{-\omega^2 I + \tilde{K}) \bar{q}_0 = 0}$$

$$\tilde{K} \bar{q}_0 = \omega^2 \bar{q}_0$$

$A \bar{x} = \lambda \bar{x}$ $\qquad$ Eigenvalue problem!

ω^2 is eigenvalue

$\bar{q}_0$ is an eigenvector

$\tilde{K}$ is guaranteed to be symmetric!

$$\tilde{K} = L^{-1} K (L^T)^{-1}$$

$$\det(A - \lambda I) = 0$$

$$\det(\tilde{K} - \omega^2 I) = 0$$

$$\tilde{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \qquad \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \begin{bmatrix} & 0 \\ 0 & \omega^2 \end{bmatrix} \right) = 0$$

$$\det\left(\begin{bmatrix} k_{11} - \omega^2 & k_{12} \\ k_{21} & k_{22} - \omega^2 \end{bmatrix} \right) = 0$$

$$(k_{11} - \omega^2)(k_{22} - \omega^2) - k_{12}k_{21} = 0 \qquad \lambda = \omega^2$$

$$\omega^4 - k_{11}\omega^2 - k_{22}\omega^2 + k_{11}k_{22} - k_{12}k_{21} = 0$$

$$\boxed{\lambda^{2} - (k_{11} + k_{22})\lambda + k_{11}k_{22} - k_{12}k_{21} = 0}$$

characteristic equation $\qquad n$: # DoF

second order polynomial in λ

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{quad eq}$$

$$\lambda = \frac{k_{11} + k_{22} \pm \sqrt{(k_{11} + k_{22})^2 - 4(k_{11}k_{22} - k_{12}k_{21})}}{2}$$

two

two eigenvalues

correspond to 4 eigen frequencies

$$\lambda = \omega^2 \Rightarrow \omega = \pm\sqrt{\lambda}$$

$$\left.\begin{array}{l} \lambda_1 : \omega_1, \omega_2 \\ \lambda_2 : \omega_3, \omega_4 \end{array}\right\} \text{natural frequencies of the system}$$

nots!

guaranteed real positive eigenvalues

Eigen vectors

$$\tilde{K}\,\bar{q}_0 = \lambda\,\bar{q}_0$$

$$\begin{bmatrix} k_{11}-\lambda_1 & k_{12} \\ k_{21} & k_{22}-\lambda_2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\tilde{K} - I\lambda)\bar{q}_0 = 0$$

2 eqs in 2 unknowns

$$\left.\begin{aligned} (k_{11}-\lambda_1)q_1 + k_{12}q_2 &= 0 \\ k_{21}q_1 + (k_{22}-\lambda_1)q_2 &= 0 \end{aligned}\right\} \text{no unique solution}$$

pick $q_2 = 1$

$$(k_{11}-\lambda_1)q_1 = -k_{12}$$

$$q_1 = \frac{-k_{12}}{k_{11}-\lambda_1} \qquad q_2 = 1$$

components of ... associated with ...

$\underline{\lambda_2}$

$$q_1 = \frac{-k_{12}}{k_{11}-\lambda_2} \qquad q_2 = 1$$

$$\bar{V}_2 = \begin{bmatrix} \dfrac{-k_{12}}{k_{11}-\lambda_2} \\ 1 \end{bmatrix}$$

$\bar{n} \times 1$

$$\bar{q}_0(t) = c_1\bar{q}_{01}\sin(\omega_1 t + \phi_1) + c_2\bar{q}_{02}\sin(\omega_2 t + \phi_2)$$

$\bar{q}_{01} = \bar{V}_1$

$\bar{q}_{02} = \bar{V}_2$

$n \times n$

$$P = \begin{bmatrix} \hat{V}_1 & \hat{V}_2 \end{bmatrix}$$

$\hat{V}_1$: unit vector aligned ...

$\hat{V}_2$: " " "

eigenvector
h λ_1

$$\bar{V}_1 = \begin{bmatrix} \frac{-k_{12}}{k_{11}-\lambda_1} \\ 1 \end{bmatrix}$$

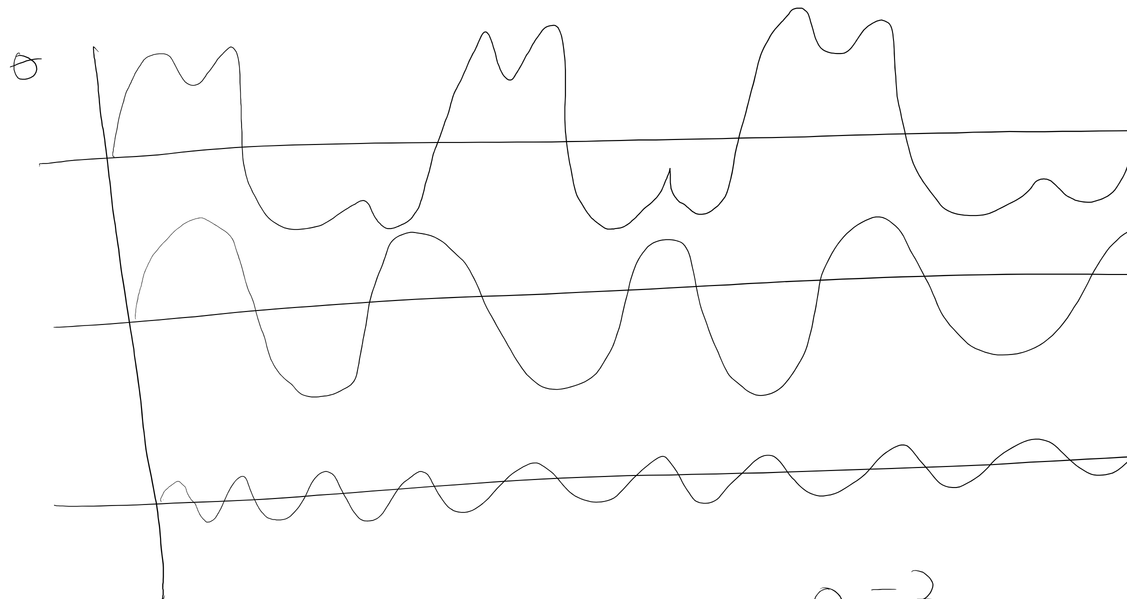
eigi vecs.
orthogonal

with $\bar{V}_1$

" $\bar{V}_2$

correspond "mode shapes"

$$\bar{q} = \begin{bmatrix}\theta\\ \phi\end{bmatrix} = c_1 \begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix} \sin(\omega_1 t + \phi_1) + C$$



$\theta_1 = 1$ $\qquad$ $\theta_2 = 2$

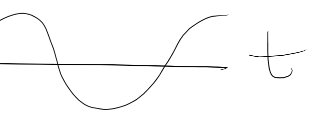
$$c_1 \begin{bmatrix}1\\ 0\end{bmatrix} \sin(\omega_1 t + \phi_1) + c_2 \begin{bmatrix}0\\ 1\end{bmatrix}$$

orthonorm

$$\bar{q} = \mathbb{P}\bar{r}$$

$$\ddot{q} + \tilde{K}\bar{q} = 0$$

$$2\begin{bmatrix}\theta_2\\ \phi_2\end{bmatrix}\sin(\omega_2 t+\phi_1)$$



$$\sin(\omega_2 t+\phi_1)$$

al set

$$P\ddot{\bar{r}} + \bar{K} P \bar{r} = 0$$

$$\underbrace{P^T P}_{I} \ddot{\bar{r}} + \underbrace{P^T \tilde{K} P}_{\Lambda} \bar{r} = 0$$

$\Lambda \rightarrow$ is

$$\ddot{\bar{r}} + \Lambda \bar{r} = 0$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\bar{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \overset{sol}{\Rightarrow} \quad \ddot{r} \quad \ddot{r}$$

guaranteed to be
a diagonal matrix

Decoupled board

$\ddot{r}_1 + \omega_1^2 r_1 = 0 \Rightarrow r_1 = A_1 \sin$

$\ddot{r}_2 + \omega_2^2 r_2 = 0 \qquad r_2 = A_2 \sin$

$$n(\omega_1 t + \phi_1)$$

$$n(\omega_2 t + \phi_2)$$

modal

coordinates